FLOW OF A GAS JET OUT OF A CHANNEL PAST A FLAT PLATE

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In this paper the solution is given to the problem of the flow of a gas jet issuing from a channel with parallel walls, and flowing past a flat plate. In the solution, use is made of a suggestion of Falkovich [1], which makes it possible to extend Chaplygin's method [2] for the solution of the gas jet problem to jet problems having a number of characteristic speeds greater than one. The solutions of the problem of the flow of a free jet past a flat plate, of the problem of a flat plate in a jet of incompressible fluid issuing from a channel, and of other problems, follow from the solution of the present problem as special cases.

1. Due to the symmetry of the problem it is sufficient to consider only half the flow (Fig. 1).

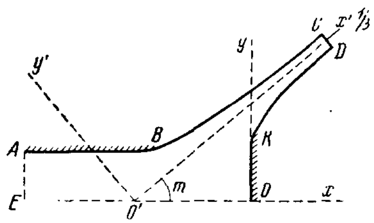

Fig. 1.

Here *AB* is the wall of the channel, *EO* is the axis of symmetry, *OK* the plate, *BC* and *DK* are free surfaces of the jet. Let v_1 be the velocity of the gas at infinity in the channel, v_2 - the velocity of the gas on the free surfaces of the jet, m - the angle of inclination of the jet velocity at infinity downstream of the plate, $2l$ - the diameter of the plate, $2d$ -

the diameter of the channel, s - the distance of the plate from the mouth of the channel. If the gas flow is denoted by Q, and $\psi = 0$ is taken on the line $EOKD$, then we must have $\psi = 1/2\ Q$ on the line ABC.

In the hodograph plane $\tau\theta$, where the variables are $\tau = v^2/v^2_{max}$, where v is the speed and v_{max} - the maximum flow speed, and θ - the angle of inclination of the velocity to the x-axis, the flow maps into a quadrant of a circle (Fig. 2).

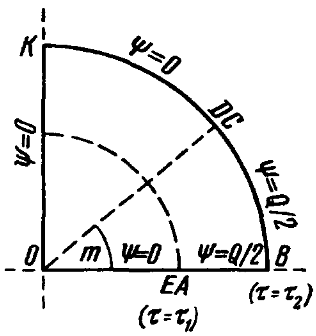

Fig. 2.

The boundary conditions are as follows:

$$\begin{aligned} &\psi = 0 \quad \text{for } 0 < \tau < \tau_1,\ \theta = 0 \\ &\psi = \tfrac{1}{2} Q \text{ for } \tau_1 < \tau < \tau_2,\ \theta = 0 \\ &\psi = 0 \quad \text{for } 0 < \tau < \tau_2,\ \theta = \tfrac{1}{2}\pi \end{aligned} \tag{1.1}$$

$$\begin{aligned} &\psi = \tfrac{1}{2} Q \text{ for } \tau = \tau_2,\ 0 < \theta < m \\ &\psi = 0 \quad \text{for } \tau = \tau_2,\ m < \theta < \tfrac{1}{2}\pi \end{aligned} \tag{1.2}$$

Assuming that the velocities are subsonic, we look for a solution of the form

$$\frac{\pi}{Q}\psi_1 = \sum_{n=1}^{\infty} a_n z_n(\tau) \sin 2n\theta \qquad \text{for } 0 < \tau < \tau_1 \tag{1.3}$$

$$\frac{\pi}{Q}\psi_2 = \frac{\pi - 2\theta}{2} + \sum_{n=1}^{\infty} \{A_n z_n(\tau) + B_n \zeta_n(\tau)\} \sin 2n\theta \text{ for } \tau_1 < \tau < \tau_2 \tag{1.4}$$

Here ψ is the stream function, $z_n(\tau)$ is an integral of Chaplygin's equation [2], regular at $\tau = 0$, and $\zeta_n(\tau)$ is another integral of that equation [1,4 [, linearly independent of the first integral.

In what follows we shall make use of the equality

$$W(z_n, \zeta_n) = \begin{vmatrix} z_n' \zeta_n' \\ z_n \zeta_n \end{vmatrix} = n\,\frac{(1-\tau)^\beta}{\tau} \qquad \left(\beta = \frac{1}{\gamma - 1}\right) \tag{1.5}$$

Here γ is the polytropic index.

The stream function ψ, defined by equations (1.3) and (1.4), satisfies the boundary condition (1.1). We now require that the boundary condition (1.2) be satisfied and that ψ_2 be the analytic continuation of ψ_1 from the region $0 < \tau < \tau_1$ into the region $\tau_1 < \tau < \tau_2$:

$$\psi_2(\tau_2) = \tfrac{1}{2} Q \text{ for } 0 < \theta < m, \qquad \psi_2(\tau_2) = 0 \qquad \text{for } m < \theta < \tfrac{1}{2}\pi \tag{1.6}$$

$$\psi_1(\tau_1) = \psi_2(\tau_1), \qquad \left[\frac{\partial \psi_1}{\partial \tau}\right]_{\tau=\tau_1} = \left[\frac{\partial \psi_2}{\partial \tau}\right]_{\tau=\tau_1} \qquad \text{for } 0 < \theta < \tfrac{1}{2}\pi \tag{1.7}$$

Substituting into (1.6) and (1.7) the expressions for ψ_1 and ψ_2, we find the coefficients a_n, A_n, B_n. This also determines the stream function ψ. We have

$$\frac{\pi}{Q}\psi_1 = \sum_{n=1}^{\infty} \frac{1}{n}\left\{-\cos 2mn + \frac{\tau_1}{n(1-\tau_1)^\beta}\left[z_n'(\tau_1)\zeta_n(\tau_2) - z_n(\tau_2)\zeta_n'(\tau_1)\right]\right\}\frac{z_n(\tau)}{z_n(\tau_2)}\sin 2n\theta \tag{1.8}$$

$$\frac{\pi}{Q}\psi_2 = \frac{\pi - 2\theta}{2} - \sum_{n=1}^{\infty}\frac{1}{n} f_n(\tau)\sin 2n\theta \tag{1.9}$$

where, for convenience of writing, we have put

$$f_n(\tau) = \cos 2mn\,\frac{z_n(\tau)}{z_n(\tau_2)} - \frac{\tau_1}{(1-\tau_1)^\beta n}\left[\zeta_n(\tau_2) z_n(\tau) - z_n(\tau_2)\zeta_n(\tau)\right]\frac{z_n'(\tau_1)}{z_n(\tau_2)} \tag{1.10}$$

For $\tau_1 = \tau_2 = \tau_0$, we find from (1.8), using (1.5), that

$$\frac{\pi}{Q}\psi = \sum_{n=1}^{\infty}\frac{1}{n}(1 - \cos 2mn)\frac{z_n(\tau)}{z_n(\tau_0)}\sin 2n\theta \tag{1.11}$$

The last result gives the stream function for the flow of a free gas jet past a flat plate, as found by Chaplygin [2].

2. Taking into account that along the plate $\psi = 0 = \text{const}$ and $\theta = 1/2\,\pi = \text{const}$, we obtain

$$dy = \frac{1}{\sqrt{2\alpha\tau}}\frac{\partial\varphi}{\partial\tau}\bigg|_{\theta=1/2\pi} d\tau = -\frac{1}{\sqrt{2\alpha\tau}}\frac{1-(2\beta+1)\tau}{2\tau(1-\tau)^{\beta+1}}\frac{\partial\psi}{\partial\theta}\bigg|_{\theta=1/2\pi} d\tau \tag{2.1}$$

Here $\sqrt{2\alpha} = v_{max}$. Putting into (2.1) the stream function ψ, as given by relations (1.8) and (1.9), and integrating from $\tau = 0$ ($y = 0$) to $\tau = \tau_2$ ($y = l$), we obtain

$$\frac{\pi}{Q} l\sqrt{2\alpha} = \sum_{n=1}^{\infty}\frac{(-1)^n 4}{4n^2 - 1}\left\{\frac{\cos 2mn}{z_n(\tau_2)(1-\tau_2)^\beta}\frac{d}{d\tau}(z_n\sqrt{\tau})_{\tau=\tau_2} - \right.$$

$$-\frac{\tau_1}{(1-\tau_1)^\beta n}\left[z_n'(\tau_1)\zeta_n(\tau_2)-z_n(\tau_2)\zeta_n(\tau_1)\right]\frac{1}{z_n(\tau_2)(1-\tau_1)^\beta}\frac{d}{d\tau}(z_n\sqrt{\tau})_{\tau=\tau_1}-$$

$$-\frac{\tau_1}{(1-\tau_1)^\beta n}\frac{z_n'(\tau_1)}{z_n(\tau_2)}\left[\zeta_n(\tau_2)\left(\frac{1}{(1-\tau_2)^\beta}\frac{d}{d\tau}(z_n\sqrt{\tau})_{\tau=\tau_2}-\frac{1}{(1-\tau_1)^\beta}\frac{d}{d\tau}(z_n\sqrt{\tau})\right)_{\tau=\tau_1}-\right.$$

$$\left.\left.-z_n(\tau_2)\left(\frac{1}{(1-\tau_2)^\beta}\frac{d}{d\tau}(\zeta_n\sqrt{\tau})_{\tau=\tau_2}-\frac{1}{(1-\tau_1)^\beta}\frac{d}{d\tau}(\zeta_n\sqrt{\tau})_{\tau=\tau_1}\right)\right]\right\}+$$

$$+\left[\frac{1}{\sqrt{\tau_1}(1-\tau_1)^\beta}-\frac{1}{\sqrt{\tau_2}(1-\tau_2)^\beta}\right] \qquad (2.2)$$

Here use has been made of the equality [2]

$$\int_0^\tau \frac{1-(2\beta+1)\tau}{\tau(1-\tau)^{\beta+1}} z_n\frac{d\tau}{\sqrt{\tau}}=\frac{4}{4n^2-1}\frac{1}{(1-\tau)^\beta}\frac{d}{d\tau}(z_n\sqrt{\tau})$$

The same expression applies to $\zeta_n(\tau)$. Use was also made of the equality

$$\int_{\tau_1}^{\tau_2}\frac{1-(2\beta+1)\tau}{\tau(1-\tau)^{\beta+1}}\frac{d\tau}{\sqrt{\tau}}=\frac{1}{\sqrt{\tau_1}(1-\tau_1)^\beta}-\frac{1}{\sqrt{\tau_2}(1-\tau_2)^\beta}$$

If the expressions $d(z_n\sqrt{\tau})/d\tau$ and $d(\zeta_n\sqrt{\tau})/d\tau$ are expanded and $\zeta_n'(\tau)$ is eliminated by means of (1.5), then we find from (2.2)

$$\frac{\pi}{Q}e\sqrt{2\alpha\tau_2}=\sum_{n=1}^{\infty}\frac{(-1)^n 2}{4n^2-1}\left\{\frac{\cos 2mn}{(1-\tau_2)^\beta}\left[1+2\tau_2\frac{z_n'(\tau_2)}{z_n(\tau_2)}\right]-\right.$$

$$\left.-\frac{1}{(1-\tau_1)^\beta}\left[\left(\frac{\tau_2}{\tau_1}\right)^{1/2}+2\tau_1\frac{z_n'(\tau_1)}{z_n(\tau_2)}\right]\right\}+\left[\left(\frac{\tau_2}{\tau_1}\right)^{1/2}\cdot\frac{1}{(1-\tau_1)^\beta}-\frac{1}{(1-\tau_2)^\beta}\right] \qquad (2.3)$$

In the same way, the functions $\zeta_n(\tau)$ are eliminated in general.

If use is made of Chaplygin's function $x_n(\tau)$ and the expression for the mass flow.

$$x_n(\tau)=\frac{\tau}{n}\frac{z_n'(\tau)}{z_n(\tau)}, \qquad Q=2d\sqrt{2\alpha\tau_1}(1-\tau_1)^\beta \qquad (2.4)$$

and also the equality

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\cos 2mn}{4n^2-1}=\frac{\pi\cos m-2}{4} \qquad (2.5)$$

then (2.3) may be written in the following form:

$$\frac{l}{d}=1-\left(\frac{\tau_1}{\tau_2}\right)^{1/2}\left(\frac{1-\tau_1}{1-\tau_2}\right)^\beta\cos m+$$

$$+\frac{2}{\pi}\left(\frac{\tau_1}{\tau_2}\right)^{1/2}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}4n}{4n^2-1}\left[\frac{z_n(\tau_1)}{z_n(\tau_2)}x_n(\tau_1)-\left(\frac{1-\tau_1}{1-\tau_2}\right)^\beta x_n(\tau_2)\cos 2mn\right] \qquad (2.6)$$

3. For what follows, we note that, from (1.10),

$$f_n(\tau_2) = \cos 2mn \tag{3.1}$$

$$f_n'(\tau_2) = \frac{n}{\tau_2}\left[\cos 2mn\, x_n(\tau_2) - \left(\frac{1-\tau_2}{1-\tau_0}\right)^\beta \frac{z_n(\tau_1)}{z_n(\tau_2)}\, x_n(\tau_1)\right] \tag{3.2}$$

Relation (3.2) is obtained by differentiating (1.10) and making use of (1.5) and (2.4).

Instead of the axes x, y, we introduce the axes x', y', taking for axis x' the straight line toward which the free surfaces of the jet tend, downstream of the plate, while for the origin O' we take the point of intersection of that straight line with the axis of symmetry (Fig. 1). Let O' have the coordinates $x = a$, $y = 0$. We shall also take the distance s as a coordinate, i.e. we shall take its sign into account. In the new axes we will have

$$\frac{\partial y'}{\partial \theta'} = \frac{1}{v(1-\tau)^\beta}\left[2\tau\frac{\partial\psi}{\partial\tau}\sin\theta' + \frac{\partial\psi}{\partial\theta'}\cos\theta'\right] \qquad (\theta' = \theta - m) \tag{3.3}$$

Integrating (3.3) from $\theta' = 0$ to $\theta' = 1/2\,\pi - m$ and putting $\tau = \tau_2$, we obtain for point M the ordinate $y' = a \sin m + l \cos m$. Integrating from $\theta' = 0$ to $\theta' = -m$ and putting $\tau = \tau_2$, we obtain for point B the ordinate $y' = -(s - a)\sin m + d\cos m$. If, from the first expression obtained in this way, we subtract the second one, complete squares and use (2.4), we obtain

$$\frac{\pi}{2}\frac{v_2(1-\tau_2)^\beta}{v_1(1-\tau_1)^\beta}\left[\frac{s}{d}\sin m + \frac{l}{d}\cos m - \cos m\right] = -\left\{\sum_{n=1}^{\infty}\frac{2\tau_2 f_n'(\tau_2)}{n}\left[\cos m\,\frac{2n(-1)^{n-1}}{4n^2-1} - \sin m\,\frac{2n}{4n^2-1}\right] + \sum_{n=1}^{\infty} 2f_n(\tau_2)\left[\cos m\,\frac{(-1)^{n-1}}{4n^2-1} - \sin m\,\frac{1}{4n^2-1}\right] + \cos m + \sin m\right\} \tag{3.4}$$

If in (3.4) we put expressions l/d, $f_n(\tau_2)$, $f_n'(\tau_2)$ from (2.6), (3.1) and (3.2), we obtain

$$\frac{s}{d} = -\left(\frac{\tau_1}{\tau_2}\right)^{1/2}\left\{\left(\frac{1-\tau_1}{1-\tau_2}\right)^\beta \sin m + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{4n}{4n^2-1}\left[\frac{z_n(\tau_1)}{z_n(\tau_2)}x_n(\tau_1) - \left(\frac{1-\tau_1}{1-\tau_2}\right)^\beta x_n(\tau_2)\cos 2mn\right]\right\} \tag{3.5}$$

Relations (2.6) and (3.5) give the dependence amongst the parameters of the problem, l/d, s/d, τ_1, τ_2 and m. As for the pressure R on the plate, it is determined quite straightforwardly from the momentum law

$$R = Qv_2\left[\frac{v_1}{v_2} - \cos m + F(\tau_1, \tau_2)\right] \tag{3.6}$$

where

$$F(\tau_1, \tau_2) = \frac{1-\tau_1}{2(\beta+1)\sqrt{\tau_1\tau_2}}\left[\left(\frac{1-\tau_2}{1-\tau_1}\right)^{\beta+1} - 1\right] \tag{3.7}$$

It is easy to confirm that for $v_{max} \to \infty$ the function $F(\tau_1, \tau_2)$ becomes the function

$$F(v_1, v_2) = \frac{1}{2}\left(\frac{v_2}{v_1} - \frac{v_1}{v_2}\right) \tag{3.8}$$

4. From the general relations obtained above, a number of particular cases follow.

Putting $\tau_1 = \tau_2 = \tau_0$, we obtain the solution to the problem of the flow of a free gas jet past a plate [2]. From (3.5) we find $s = -\infty$, as it must be. Equations (2.6) and (3.6) will give

$$\frac{l}{d} = 1 - \cos m + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}4n}{4n^2-1}(1-\cos 2mn)\,x_n(\tau_0), \qquad R = Qv_0(1-\cos m)$$

Putting $m = 1/2\,\pi$, we obtain the solution to the problem of a gas jet issuing from a channel and striking an infinite plate. We note that this problem may also be considered as the problem of the impact of two jets issuing from two similar and symmetrically placed channels.

The relative position of channel and plate shown in Fig. 1 corresponds to the case $s < 0$. For $s > 0$ we will have the solution to the problem of flow over a plate which is inside the channel at a finite distance from the opening.

In particular, for $m = 0$, we will have the solution to the flow past a plate in a channel whose opening is infinitely far away. This last problem, as was already noted by Zhukovskii [3], is equivalent to the problem of flow out of a vessel of finite width and infinite length, which, for the case of a gas was solved by Falkovich [2]. The method of Falkovich was used, in fact, in the solution of the present problem.

Putting $\tau_1 = \tau_2 = 0$ in (2.6), (3.5), (3.6), we obtain the solution to the problem of an incompressible fluid issuing from a channel and flowing past a plate. In this case the summation of the series is easy. The result is

$$\frac{l}{d} = 1 - \frac{v_1}{v_2}\cos m + \frac{2}{\pi}\left[\sin m \ln \operatorname{tg}\left(\frac{\pi}{4} + \frac{m}{2}\right) + \left(\frac{v_1}{v_2} - \frac{v_2}{v_1}\right)\operatorname{arc\,tg}\frac{v_1}{v_2}\right]$$

$$\frac{s}{d} = -\frac{v_1}{v_2}\left\{\sin m + \frac{2}{\pi}\left[\cos m \ln \operatorname{tg}\frac{m}{2} + \left(\frac{v_1}{v_2} + \frac{v_2}{v_1}\right)\operatorname{Ar\,th}\frac{v_1}{v_2}\right]\right\}$$

$$R = Qv_2\left[\frac{1}{2}\left(\frac{v_2}{v_1} + \frac{v_1}{v_2}\right) - \cos m\right]$$

If in these last equations we put

$$\frac{v_1}{v_2} = \operatorname{tg}\frac{\nu}{2}$$

they can be easily obtained in the form given by Zhukovskii[3].

BIBLIOGRAPHY

1. Falkovich, S.V., K teorii gazov'ikh strui (On the theory of gas jets). *PMM* Vol. 21, No. 4, pp. 459-464, 1957.

2. Chaplygin, S.A., *O gazov'ikh struyakh (On Gas Jets)*. GTTI, Moscow-Leningrad, 1949.

3. Zhukovskii N.E., Vidoizmenenie metoda Kirkhgoffa (Modification of Kirchhoff's method). *Coll. Works*, Vol. 1, GITTL, Moscow-Leningrad, 1948.

4. Cherry, T.M., Asymptotic expansions for the hypergeometric functions occurring in gas-flow theory. *Proc. roy. Soc.*, London, Ser. A, Vol. 202, 1950.

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